

# Test particle description of transport processes for states with a continuous mass spectrum

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Aiming at a description of transport processes where the dynamically generated width of the states is potentially large a transport equation beyond the quasiparticle approximation is derived in first order gradient expansion. An effective particle number is identified which is exactly conserved by the coarse grained transport equation. Using a test particle ansatz for this conserved quantity allows to rewrite the transport equation into equations of motion for test particles. The two-body collision terms are formulated in terms of the test particles which gain non-trivial renormalization factors due to the coarse graining process.

## 1 Motivation

This talk is based on my recent work<sup>1</sup>. Due to limited space I refer to this work<sup>1</sup> concerning more details and references.

For the description of heavy-ion collisions the following Boltzmann-type equation is commonly used:

$$\left( \partial_t + \frac{\vec{p}}{m} \vec{\nabla}_x - \vec{\nabla}_x U(t, \vec{x}) \vec{\nabla}_p \right) f(t, \vec{x}, \vec{p}) = I_{\text{coll}} \quad (1)$$

where the l.h.s. contains the drift and the Vlasov term and the r.h.s. incorporates the collisions. The central quantity here is the *on-shell* phase space distribution  $f$ . “On-shell” simply means that the energy is not an additional free variable on which  $f$  might depend but is instead given by the appropriate mass-shell condition. Therefore  $f$  depends only on the three-momentum (and the space-time variables). I will discuss in a moment why this might be an unpleasant restriction for various processes relevant for the description of heavy-ion reactions. Before I come to that point I want to discuss briefly how eq. (1) can be solved: One usually involves a test particle ansatz

$$f(t, \vec{x}, \vec{p}) \sim \sum_i \delta^{(3)}(\vec{x} - \vec{x}_i(t)) \delta^{(3)}(\vec{p} - \vec{p}_i(t)) \quad (2)$$

and deduces from (1) the collisions of these test particles and the evolution of their coordinates  $\vec{x}_i$ ,  $\vec{p}_i$  between collisions. In what follows I will concentrate on the evolution part. The equations of motion for the test particle coordinates

between collisions are obtained by inserting (2) in (1) and replacing the collision term by zero. In this way one obtains the equation

$$\sum_i \left( \left[ -\dot{\vec{x}}_i + \frac{\vec{p}_i}{m} \right] \vec{\nabla}_x + \left[ -\dot{\vec{p}}_i - \vec{\nabla}_{x_i} U(t, \vec{x}_i) \right] \vec{\nabla}_p \right) \delta^{(3)}(\vec{x} - \vec{x}_i) \delta^{(3)}(\vec{p} - \vec{p}_i) = 0.$$

This equation can be fulfilled by demanding that both expressions in square brackets have to vanish independently for every test particle  $i$ . In this way one gets the familiar Newtonian equations for the test particle coordinates

$$\dot{\vec{x}}_i = \frac{\vec{p}_i}{m}, \quad \dot{\vec{p}}_i = -\vec{\nabla}_{x_i} U(t, \vec{x}_i).$$

From a practical point of view one expects to get a sufficiently correct expression for the phase space density if the number of test particles is reasonably large. This traditional transport theoretical framework is capable to describe states with energies close to or at the mass-shell point. For heavy-ion collisions at sufficiently high energy, however, broad resonances come into play with decay widths of the order of the typical kinetic energies. In addition, also the width due to collisional broadening might become comparably large. In this case the quasi-particle approximation which enters the field theoretical derivation of the Boltzmann equation (1) becomes questionable and one has to find extensions of that model. The new central quantity of interest will be the Wigner transform (see below) of the two-point function denoted by  $S^<(t, \vec{x}; p_0, \vec{p})$ . Note that this quantity depends explicitly on the energy and not only on the three-momentum. It is the purpose of the next section to derive a transport equation for  $S^<$  *without* resorting to the quasi-particle approximation. Then I will discuss how the obtained transport equation can be solved by an appropriate test particle ansatz. One of the questions will be for which quantity a test particle ansatz is justified. From the discussion outlined above it is already now apparent that in such an extended test particle ansatz the energy has to be treated as an independent variable. Thus, this ansatz for a so far unspecified quantity has the form

$$\tilde{S}^<(t, \vec{x}; p) \sim \sum_i \delta^{(3)}(\vec{x} - \vec{x}_i(t)) \delta(p_0 - E_i(t)) \delta^{(3)}(\vec{p} - \vec{p}_i(t)). \quad (3)$$

## 2 From the Kadanoff-Baym equations to the transport equation

It is obviously appropriate for a talk at this workshop to start out from the Kadanoff-Baym equations:

$$\left( i \frac{\partial}{\partial t_1} + \frac{\Delta_1}{2m} \right) D^<(1, 1') = \int d\bar{1} \left[ \Sigma^{\text{ret}}(1, \bar{1}) D^<(\bar{1}, 1') + \Sigma^<(1, \bar{1}) D^{\text{av}}(\bar{1}, 1') \right] \quad (4)$$

$$\left(i\frac{\partial}{\partial t_1} + \frac{\Delta_1}{2m}\right) D^>(1, 1') = \int d\bar{1} [\Sigma^{\text{ret}}(1, \bar{1}) D^>(\bar{1}, 1') + \Sigma^>(1, \bar{1}) D^{\text{av}}(\bar{1}, 1')] \quad (5)$$

$$\left(i\frac{\partial}{\partial t_1} + \frac{\Delta_1}{2m}\right) D^{\text{ret}}(1, 1') = \delta^{(4)}(1, 1') + \int d\bar{1} \Sigma^{\text{ret}}(1, \bar{1}) D^{\text{ret}}(\bar{1}, 1') \quad (6)$$

with

$$\begin{aligned} iD^<(x, y) &= \pm \langle \psi^\dagger(y) \psi(x) \rangle, & iD^>(x, y) &= \langle \psi(x) \psi^\dagger(y) \rangle, \\ D^{\text{ret}}(x, y) &= \Theta(x_0 - y_0) [D^>(x, y) - D^<(x, y)], \\ D^{\text{av}}(x, y) &= \Theta(y_0 - x_0) [D^<(x, y) - D^>(x, y)]. \end{aligned}$$

The plus/minus sign refers to bosons and fermions, respectively. Actually all information of this non-relativistic system is already contained in the first two equations (4) and (5). In what follows it is more convenient however to additionally deal with the equation (6) for the retarded propagator. As usual for the derivation of kinetic equations from the underlying field theoretical equations I assume now that, concerning the transport phenomena one wants to describe, the dependence of all two-point functions and self-energies on their respective (macroscopic) center-of-mass variables  $X = (x+y)/2$  is much weaker than the dependence on the (microscopic) difference variable  $u = x - y$ . In this case it is useful to perform a Fourier transformation of all quantities with respect to their respective difference variable (Wigner transformation)

$$\bar{D}^<(X, p) = \int d^4u e^{ipu} D^<(X + u/2, X - u/2),$$

expand in gradients with respect to  $X$  and neglect all terms which contain more than one derivative in  $X$ . I also introduce the generalized Poisson bracket

$$[A, B] = \partial_{X_0} A \partial_{p_0} B - \partial_{p_0} A \partial_{X_0} B - \vec{\nabla}_X A \vec{\nabla}_p B + \vec{\nabla}_p A \vec{\nabla}_X B.$$

By the procedure outlined above the Kadanoff-Baym equations (4-6) turn into three (gradient expanded) complex-valued equations. These equations can be separated into their real and imaginary parts by introducing the following real-valued quantities

$$\begin{aligned} S^<(X, p) &= \pm i \bar{D}^<(X, p), \\ \mathcal{A}(X, p) &= -2\text{Im} \bar{D}^{\text{ret}}(X, p) = i[\bar{D}^>(X, p) - \bar{D}^<(X, p)]. \end{aligned}$$

From the Kadanoff-Baym equations one gets by gradient expansion the following set of six (real-valued) equations:

$$\left(p_0 - \frac{\vec{p}^2}{2m} - \text{Re} \bar{\Sigma}^{\text{ret}}\right) S^< = \pm i \bar{\Sigma}^< \text{Re} \bar{D}^{\text{ret}} - \frac{1}{4}[\Gamma, S^<] + \frac{1}{4}[\pm i \bar{\Sigma}^<, \mathcal{A}], \quad (7)$$

$$[p_0 - \frac{\vec{p}^2}{2m} - \text{Re}\bar{\Sigma}^{\text{ret}}, S^<] = \Gamma S^< \mp i\bar{\Sigma}^< \mathcal{A} + [\pm i\bar{\Sigma}^<, \text{Re}\bar{D}^{\text{ret}}], \quad (8)$$

$$\left(p_0 - \frac{\vec{p}^2}{2m} - \text{Re}\bar{\Sigma}^{\text{ret}}\right) \mathcal{A} = \Gamma \text{Re}\bar{D}^{\text{ret}}, \quad (9)$$

$$[p_0 - \frac{\vec{p}^2}{2m} - \text{Re}\bar{\Sigma}^{\text{ret}}, \mathcal{A}] = [\Gamma, \text{Re}\bar{D}^{\text{ret}}], \quad (10)$$

$$\left(p_0 - \frac{\vec{p}^2}{2m} - \text{Re}\bar{\Sigma}^{\text{ret}}\right) \text{Re}\bar{D}^{\text{ret}} = 1 - \frac{1}{4}\Gamma \mathcal{A}, \quad (11)$$

$$[p_0 - \frac{\vec{p}^2}{2m} - \text{Re}\bar{\Sigma}^{\text{ret}}, \text{Re}\bar{D}^{\text{ret}}] = -\frac{1}{4}[\Gamma, \mathcal{A}] \quad (12)$$

where I have introduced the width  $\Gamma = i(\bar{\Sigma}^> - \bar{\Sigma}^<) = i(\bar{\Sigma}^{\text{ret}} - \bar{\Sigma}^{\text{av}})$ . In the end I am aiming at decoupled equations for the generalized phase space distribution  $S^<$ , the spectral function  $\mathcal{A}$ , and the real part of the retarded propagator  $\text{Re}\bar{D}^{\text{ret}}$ . Obviously one has twice as much equations as quantities which one wants to determine. Some of these equations must be redundant. From the purely algebraic equations (9,11) one immediately obtains

$$\mathcal{A}(X, p) = \frac{\Gamma(X, p)}{\left(p_0 - \frac{\vec{p}^2}{2m} - \text{Re}\bar{\Sigma}^{\text{ret}}(X, p)\right)^2 + \frac{1}{4}\Gamma^2(X, p)}, \quad (13)$$

$$\text{Re}\bar{D}^{\text{ret}}(X, p) = \frac{p_0 - \frac{\vec{p}^2}{2m} - \text{Re}\bar{\Sigma}^{\text{ret}}(X, p)}{\left(p_0 - \frac{\vec{p}^2}{2m} - \text{Re}\bar{\Sigma}^{\text{ret}}(X, p)\right)^2 + \frac{1}{4}\Gamma^2(X, p)}. \quad (14)$$

These expressions automatically solve also eqs. (10,12). In the quasi-particle regime, i.e. for  $\Gamma \rightarrow 0$ , the remaining two equations (7,8) turn into the mass-shell constraint and the transport equation, respectively. For our case of arbitrary width  $\Gamma$ , however, these two equations contain the same information provided one realizes by inspection of (8) that the combination  $\pm i\bar{\Sigma}^< - \frac{\Gamma}{\mathcal{A}}S^<$  is effectively a first order gradient. Since all equations are exact up to (including) first order gradients the following replacement on the r.h.s. of (7,8) is allowed:

$$[\pm i\bar{\Sigma}^<, \dots] \approx \left[\frac{\Gamma}{\mathcal{A}}S^<, \dots\right]. \quad (15)$$

After this replacement (7) and (8) become identical. Using (13,14) one finally gets the desired transport equation for  $S^<$ :

$$\begin{aligned} \frac{1}{2}\Gamma \mathcal{A} [p_0 - \frac{\vec{p}^2}{2m} - \text{Re}\bar{\Sigma}^{\text{ret}}, S^<] - \frac{1}{2}\mathcal{A} [\Gamma, (p_0 - \frac{\vec{p}^2}{2m} - \text{Re}\bar{\Sigma}^{\text{ret}})S^<] \\ = \Gamma S^< \mp i\bar{\Sigma}^< \mathcal{A} (= i\bar{\Sigma}^> S^< \mp i\bar{\Sigma}^< S^>) . \end{aligned} \quad (16)$$

Before I will point out how this equation can be solved by a test particle ansatz I seemingly change my topic and discuss for which quantities a test particle ansatz makes sense and for which quantities it does not.

### 3 Test particles I: A simple example

Imagine the following situation: One has a theory describing the evolution of a quantity  $f(t, p)$ . This theory, however, is too complicated to solve it exactly. (This is presumably easy to imagine.) The only thing one knows about that theory is that it conserves a particle number  $N(t) = \int dp f(t, p)$ . By an approximation scheme one obtains from the exact theory the following approximate (transport) equation (which resembles a Vlasov equation):

$$\frac{\partial}{\partial t} [(1 - \kappa(t, p))f(t, p)] + F(t) \frac{\partial}{\partial p} f(t, p) = 0 \quad (17)$$

with a force term  $F(t)$  and a renormalization  $\kappa$  which both are not further specified. Obviously this approximate (transport) equation does *not* conserve the particle number  $N$  but instead the quantity  $\tilde{N}(t) = \int dp (1 - \kappa(t, p))f(t, p)$ . I now discuss two approaches to solve (17), namely test particle ansätze for  $f$  and for  $\tilde{f} = (1 - \kappa)f$ , respectively: In general, a test particle representation for any of the two quantities is given by

$$\left. \begin{matrix} f \\ \text{or} \\ \tilde{f} \end{matrix} \right\} = \frac{1}{L} \sum_{i=1}^M \delta(p - p_i(t)) \quad (18)$$

where  $L$  is a normalization constant and  $M$  the number of test particles. Obviously the momentum integral over (18) is conserved and given by  $M/L$ . Therefore, a test particle ansatz for  $f$  to solve (17) seems to be inappropriate since the momentum integral over  $f$  (which yields  $N$ ) is not conserved by (17). Nonetheless, it is instructive to figure out which equations one gets with a test particle ansatz for  $f$ . In this case one finds

$$0 = \sum_i \left\{ -\frac{d\kappa(t, p_i(t))}{dt} + \left[ F(t) - [1 - \kappa(t, p_i(t))] \dot{p}_i(t) \right] \frac{\partial}{\partial p} \right\} \delta(p - p_i(t)).$$

This yields *two* equations of motion for *one* quantity:

$$\begin{aligned} \frac{d\kappa}{dt} = \frac{\partial \kappa}{\partial t} + \dot{p}_i \frac{\partial \kappa}{\partial p_i} = 0 & \Rightarrow \dot{p}_i = -\frac{\frac{\partial \kappa}{\partial t}}{\frac{\partial \kappa}{\partial p_i}}, \\ F = \dot{p}_i(1 - \kappa) & \Rightarrow \dot{p}_i = \frac{F}{1 - \kappa}, \end{aligned}$$

i.e. an in general overdetermined system. This is the consequence of the fact that the integral over  $f$ , i.e.  $N$ , is not conserved by the transport equation (17). Instead a test particle ansatz for  $\tilde{f} = (1 - \kappa)f$  yields

$$0 = \sum_i \left[ \frac{F(t)}{1 - \kappa(t, p_i(t))} - \dot{p}_i(t) \right] \frac{\partial}{\partial p} \delta(p - p_i(t)),$$

i.e. *one* equation of motion

$$\dot{p}_i(t) = \frac{F(t)}{1 - \kappa(t, p_i)}.$$

Obviously this is a very intricate situation: by construction  $N$  is conserved by the exact theory but not by the approximate transport equation. Therefore a test particle ansatz for  $f$  does not make sense. Thus I propose the following strategy: Solve the transport equation by a test particle ansatz for  $\tilde{f}$  and calculate from the solution the expression  $dN/dt$ . Since this expression should vanish in the exact theory it provides a test for the accuracy of the approximation scheme which has led to the transport equation (17). The lesson from this simple example is the following: To solve a transport equation by a test particle ansatz one has to find a quantity which is *exactly* conserved by the approximate transport equation. It may appear that this quantity does not coincide with the one which is exactly conserved by the exact (quantum field) theory.

#### 4 Test particles II: From the transport equation to the equations of motion for test particles

For the full quantum field theory given by the Kadanoff-Baym equations (4-6) the corresponding particle number is given by

$$N(t) = \int \frac{d^3x d^4p}{(2\pi)^4} S^<(t, \vec{x}; p).$$

For appropriately chosen self-energies (“ $\Phi$ -derivable”) this quantity is conserved. I will restrict myself to the commonly used two-body collision terms (Born terms) which indeed conserve  $N$ . Here e.g. the out-rate is given by

$$\begin{aligned} i\bar{\Sigma}^>(X, p) &= \int \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \frac{d^4p_3}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p + p_1 - p_2 - p_3) \\ &\times \frac{1}{2} (\bar{v}(\vec{p} - \vec{p}_2) \pm \bar{v}(\vec{p} - \vec{p}_3))^2 S^<(X, p_1) S^>(X, p_2) S^>(X, p_3). \end{aligned} \quad (19)$$

If the quantity  $N$  was also conserved by the transport equation (16) a test particle ansatz for  $S^<$  would make sense. However, one gets

$$\frac{d}{dt}N(t) = \frac{d}{dt} \int d^3x \int \frac{d^4p}{(2\pi)^4} S^< K + \int d^3x \int \frac{d^4p}{(2\pi)^4} \frac{2}{\Gamma \mathcal{A}} (\pm i \bar{\Sigma}^< S^> - i \bar{\Sigma}^> S^<) \quad (20)$$

with

$$K(X, p) = \frac{\partial \text{Re} \bar{\Sigma}^{\text{ret}}(X, p)}{\partial p_0} + \frac{p_0 - \frac{\vec{p}^2}{2m} - \text{Re} \bar{\Sigma}^{\text{ret}}(X, p)}{\Gamma(X, p)} \frac{\partial \Gamma(X, p)}{\partial p_0}.$$

I have not found any reason why the r.h.s. of (20) should vanish and I strongly conjecture that it does not. (Actually it is hard to *prove* that a quantity is *not* conserved.) This conjecture is supported by the finding that a closely related quantity, namely

$$\tilde{N}(t) = \int \frac{d^3x d^4p}{(2\pi)^4} \tilde{S}^<(t, \vec{x}; p)$$

with  $\tilde{S}^< = \frac{1}{2} \Gamma \mathcal{A} S^< (1 - K)$  is conserved by the transport equation (16):

$$\frac{d}{dt} \tilde{N}(t) = \int d^3x \int \frac{d^4p}{(2\pi)^4} (\pm i \bar{\Sigma}^< S^> - i \bar{\Sigma}^> S^<) = 0$$

where the last equation holds for the Born approximation to the self-energies. I think it is hard to conceive that both  $N$  and  $\tilde{N}$  are exactly conserved by the transport equation. Assuming that  $N$  is not conserved a test particle ansatz for  $S^<$  is inappropriate. Instead, from the test particle ansatz (3) for  $\tilde{S}^<$  which is the density corresponding to the conserved *effective* particle number  $\tilde{N}$  one deduces the equations of motion for test particles *between two collisions*:

$$\begin{aligned} \dot{E}_i &= \frac{1}{1-K} \left( \partial_t \text{Re} \bar{\Sigma}^{\text{ret}} + \frac{\Delta E_i}{\Gamma} \partial_t \Gamma \right), \\ \dot{\vec{x}}_i &= \frac{1}{1-K} \left( \frac{\vec{p}_i}{m} + \vec{\nabla}_{p_i} \text{Re} \bar{\Sigma}^{\text{ret}} + \frac{\Delta E_i}{\Gamma} \vec{\nabla}_{p_i} \Gamma \right), \\ \dot{\vec{p}}_i &= -\frac{1}{1-K} \left( \vec{\nabla}_{x_i} \text{Re} \bar{\Sigma}^{\text{ret}} + \frac{\Delta E_i}{\Gamma} \vec{\nabla}_{x_i} \Gamma \right) \end{aligned}$$

with the off-shellness  $\Delta E_i(t, \vec{x}_i; E_i, \vec{p}_i) = E_i - \frac{\vec{p}_i^2}{2m} - \text{Re} \bar{\Sigma}^{\text{ret}}(t, \vec{x}_i; E_i, \vec{p}_i)$ . The obviously new ingredient as compared to the traditional quasi-particle approximation is this off-shellness. Its evolution in time can be obtained from the equations of motion given above to be

$$\frac{d}{dt} \Delta E_i = \frac{\Delta E_i}{\Gamma} \frac{d}{dt} \Gamma. \quad (21)$$

One apparent merit of this equation is the fact that test particles automatically get back to their mass-shell if the width goes to zero. I note in passing that a crucial step to obtain (21) consists in the replacement (15). Finally I want to comment on the Born terms which in a test particle picture describe the collisions between these test particles. Obviously e.g. the out-rate (19) has to be rewritten in terms of test particle distributions:

$$i\bar{\Sigma}^>(X, p) = \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \frac{d^4 p_3}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p + p_1 - p_2 - p_3) \frac{1}{2} (\bar{v}(\vec{p} - \vec{p}_2) \pm \bar{v}(\vec{p} - \vec{p}_3))^2 \\ \times \frac{\tilde{S}_1^<}{\frac{1}{2}\Gamma_1 \mathcal{A}_1 (1 - K_1)} \left( \mathcal{A}_2 \pm \frac{\tilde{S}_2^<}{\frac{1}{2}\Gamma_2 \mathcal{A}_2 (1 - K_2)} \right) \left( \mathcal{A}_3 \pm \frac{\tilde{S}_3^<}{\frac{1}{2}\Gamma_3 \mathcal{A}_3 (1 - K_3)} \right)$$

with  $\tilde{S}_j^< = \tilde{S}^<(X, p_j)$  etc. The fact that  $\tilde{S}^<$  instead of  $S^<$  is represented by test particles causes rather non-trivial modifications for the collision integrals. In turn, disregarding these modifications amounts to solving a transport equation which might be rather different from the one which one actually wants to solve.

## 5 Test particles III: Comparison of the test particle method and Klimontovich's approach to kinetic theory

Before summarizing I want to comment on the strong formal similarity of the test particle ansatz (2) to the microscopic phase space density

$$N_M(\vec{x}, \vec{p}, t) = \sum_{i=1}^N \delta^{(3)}(\vec{x} - \vec{x}_i(t)) \delta^{(3)}(\vec{p} - \vec{p}_i(t)) \quad (22)$$

introduced by Klimontovich<sup>2</sup> for a classical  $N$ -particle system. I first will briefly outline in which context Klimontovich uses this function: Aiming at a statistical description of a classical system of  $N$  interacting particles the quantity (22) is introduced as the phase space number density of a specific microscopic realization of the macroscopic system. Assuming that the particles interact via central two-body forces obtained from the potential  $\Phi(|\vec{x}_i - \vec{x}_j|)$  one obtains Newton's equations for the coordinates of particle  $i$ :

$$\dot{\vec{x}}_i = \frac{\vec{p}_i}{m}, \quad \dot{\vec{p}}_i = -\vec{\nabla}_{x_i} \sum_j \Phi(|\vec{x}_i - \vec{x}_j|). \quad (23)$$

Combined with the equation of continuity in phase space  $dN_M/dt = 0$  one gets

$$\partial_t N_M + \frac{\vec{p}}{m} \vec{\nabla}_x N_M - \vec{\nabla}_x \int d^3 x' d^3 p' \Phi(|\vec{x} - \vec{x}'|) N_M(\vec{x}', \vec{p}', t) \vec{\nabla}_p N_M = 0 \quad (24)$$



called the Klimontovich equation. Formally this equation resembles the Vlasov equation. However, there is an important difference between the Klimontovich and the Vlasov equation: The latter is an approximate equation for the ensemble averaged one-particle distribution function  $f$  (which can be obtained e.g. from (1) by neglecting the collision term  $I_{\text{coll}}$ ). In contrast, no statistical information has entered (24) yet. There, one microscopic realization of the  $N$ -particle system is considered. The crucial step which involves statistics amounts to average (24) over all microscopic realizations which form the macroscopic state, i.e. to consider an average weighted by the  $N$ -particle phase space distribution function. In this way one deduces from (24) an equation of motion for the ensemble averaged quantity  $\bar{N}_M$  which coincides (up to normalization) with the one-particle phase space density  $f$ . Obviously in the resulting equation the average over the product of two microscopic phase space densities appears (stemming from the two-body interaction part). From (24) one can also deduce an equation of motion for such a product which in turn involves products of three microscopic phase space densities. In this way one generates a hierarchy of equations which is analogous to the BBGKY hierarchy of  $n$ -particle distribution functions. It is the merit of Klimontovich's method that the whole hierarchy can be deduced from one equation (24). If in the ensemble average of (24) the average of the product of two microscopic phase space densities is approximated by the product of averages then one obtains the Vlasov equation. In this case all possible correlations are neglected. If such correlations are retained one can obtain — using a proper approximation scheme — e.g. collision terms of Boltzmann type. To summarize, in Klimontovich's method the microscopic phase space density (22) is introduced to obtain (24) from which all equations of motion for ensemble averaged products of these densities can be deduced. Within proper approximation schemes the well-known (ensemble averaged) kinetic equations with and without collision terms can be obtained. Now one might turn the procedure around: Starting from (24) one might try to find a proper way to solve this equation. This can be achieved by the ansatz (22) which one might be tempted to call “test particle ansatz”. (This, however, is misleading.) One obtains in this way the equations of motion (23). Solving these equations e.g. by a numerical simulation yields indeed *one* solution of (24). This, however, is not the whole story. One is actually interested in the ensemble averaged  $n$ -particle distribution functions. To obtain these one has to solve (23) for various initial conditions which belong to the same macroscopic state. Calculating averages and fluctuations of the various simulations (characterized by the initial conditions) yields the quantities of interest. To compare this strategy to the test particle ansatz discussed in the previous sections one has to realize that it is the purpose of the latter

to yield a practicably applicable method to solve the *ensemble averaged* Boltzmann equation and not a microscopic equation. Thus one does not start with arbitrary initial conditions for the test particle coordinates but instead with a large number of test particles such that the initial one-particle phase space distribution is reasonably well reproduced by the test particle distribution. A second difference concerns the collision terms, i.e. dissipative terms which are explicitly present in the Boltzmann equation (1) but not in the microscopic Klimontovich equation (24). While the equations of motion for the test particles *between collisions* are — for a classical system — indeed identical to the microscopic equations (23) the test particles in addition scatter on each other with the respective scattering cross sections which enter the Boltzmann collision terms. To summarize, in spite of the strong formal similarity between the test particle ansatz (2) and Klimontovich’s microscopic phase space density (22) there are important conceptual differences in the two approaches. This becomes even more pronounced if genuine quantum corrections like the off-shell effects discussed above are included in the test particle approach.

## 6 Summary

I have presented a scheme to derive from the Kadanoff-Baym equations a transport equation beyond the quasi-particle approximation for states with potentially large width. I have argued that the particle number  $N$  defined for the full quantum field theory is not exactly conserved by the transport equation. Instead it can be shown that the effective particle number  $\tilde{N}$  is conserved. I have derived equations of motion for the test particle coordinates from a test particle ansatz for  $\tilde{S}^<$ . Finally I have presented the modifications of the collision integrals once they are used to describe collisions between the test particles. Once the transport equation is solved by this ansatz one can calculate  $dN/dt$  and find out to which amount the conservation of  $N$  is violated. This provides a test for the accuracy of the gradient expansion.

**Acknowledgments:** I would like to thank the organizers and especially Michael Bonitz for providing the basis for an interesting and inspiring workshop.

## References

1. S. Leupold, nucl-th/9909080, submitted to **Nucl. Phys. A**.
2. Yu. L. Klimontovich, Sov. Phys. JETP, **6**, 753 (1958); **35**, 920 (1972); The statistical theory of non-equilibrium processes in a plasma, Pergamon Pr., Oxford, 1967.